Journal of Pure and Applied Algebra 26 (1982) 189–190
North-Holland Publishing Company

VALUATION RINGS AND SIMPLE TRANSCENDENTAL FIELD EXTENSIONS

William J. HEINZER*
Department of Mathematics, Purdue University, W. Lafayette, IN 47907, USA

Communicated by H. Bass
Received 17 September 1981

If a valuation ring V on a simple transcendental field extension $K_0(X)$ is such that the residue field k of V is not algebraic over the residue field k_0 of $V_0 = V \cap K_0$, then for k_0 a perfect field it is shown that k is obtained from k_0 by a finite algebraic followed by a simple transcendental field extension.

1980 Mathematics Subject Classification: Primary 13A18, 12F20.

Key words and phrases: Valuation ring, Ruled field extension.

Let $K = K_0(X)$ be a simple transcendental field extension, let V be a valuation ring on K, and let $V_0 = V \cap K_0$. If the residue field k of V is not algebraic over the residue field k_0 of V_0, then Nagata showed in [3] that k is ruled over k_0 when V is rank n discrete, and asked if this is the case for arbitrary V. J. Ohm in [5] has recently answered this question affirmatively when k_0 is of characteristic zero. We present a shorter but less elementary proof of this result that also works for k_0 a perfect field of positive characteristic. The question remains open for imperfect k_0.

Let $G_0 \subset G$ denote the value groups of $V_0 \subset V$. The assumption that k/k_0 is not algebraic implies that G_0 is of finite index in G [2, Corollary 1, p. 166]. By replacing X if necessary by X^{-1}, we may assume that $X \in V$. Let $D = K_0[X] \cap V$, and let P denote the contraction to D of the maximal ideal of V. The following lemma implies that $D_P = V$. The basic idea in the proof of the lemma comes from [4, Theorem 5.1, p. 330].

Lemma. *Let $F \subset K$ be fields, let V be a valuation ring on K, let $W = V \cap F$, and let Λ be a subset of V such that the ring $F[\Lambda]$ has quotient field K. If the value groups $H \subset G$ of $W \subset V$ are such that G/H is a torsion group, then V is a localization of $D = V \cap F[\Lambda]$.*

Proof. We have $W[\Lambda] \subset D$, so D has quotient field K. Hence if $y \in V$, $y = a/b$ with

* Research supported by NSF Grant 80-02201.

$a, b \in D$, $b \neq 0$. Since G/H is a torsion group, there exists a positive integer n and an element c of W such that $b^n V = cV$. Thus, $b^n/c \in F[\Lambda] \cap V = D$. Moreover, if P is the contraction to D of the maximal ideal of V, then $b^n/c \notin P$. We have $(b^n/c)y = b^{n-1}a/c \in F[\Lambda] \cap V = D$, so $y \in D_P$. Therefore $D_P = V$. □

We note that by permutability of localization and residue class formation the fact that $D_P = V$ implies that D/P has quotient field k. We can state our result as follows.

Theorem. *There exists a finite algebraic field extension k_0' of k_0 and an element t transcendental over k_0' such that $k_0 < D/P \subset k_0'[t]$. If k_0 is assumed to be a perfect field, then the integral closure of D/P is a polynomial ring $l[u]$ where l is a field between k_0 and k_0'. In particular, when k_0 is perfect, k is ruled over k_0.*

Proof. Since k/k_0 is not algebraic, it is a finitely generated field extension [2, Corollary 1, p. 166]. Hence to show that there exists k_0' finite algebraic over k_0 with $k_0 < D/P \subset k_0'[t]$, it will suffice to show that if $\bar{k}_0$ is an algebraic closure of k_0, then $k_0 < D/P \subset \bar{k}_0[t]$. We argue as in [3, p. 91]. Let K_0' be an algebraic closure of K_0 and let V' be an extension of V to $K_0'(X)$. Let $V_0' = V' \cap K_0'$, and set $D' = K_0'[X] \cap V'$. We note that $D' \cap K = D$, and if P' denotes the contraction to D' of the maximal ideal of V', then $P' \cap D = P$ and $D'_{P'} = V'$. We wish to show that $D' = V_0'[aX + b]$ for some $a, b \in K_0'$. There exists $y \in D$ such that the residue of y in D/P is transcendental over k_0. We have $y = \prod_{i=1}^{n} (a_i X + b_i)$ in $K_0'[X]$, and as noted by Nagata in [3, p. 91], using that the value group of V_0' is divisible, we may assume that each $a_i X + b_i$ is a unit of V'. Hence the residue of some $a_i X + b_i$ is transcendental over $\bar{k}_0$, and for this i we have $D' = V_0'[a_i X + b_i]$. It follows that $D'/P' = \bar{k}_0[t]$ where t is the residue of $a_i X + b_i$. This gives $k_0 < D/P \subset \bar{k}_0[t]$, and hence $k_0 < D/P \subset k_0'[t]$ for k_0' a finite algebraic extension of k_0. If k_0 is perfect, then [1, (2.9), p. 322] implies that the integral closure of D/P is a polynomial ring $l[u]$ where l is a field between k_0 and k_0'. Hence k is ruled over k_0.

References

[1] S. Abhyankar, P. Eakin, and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring, J. of Algebra 23 (1972) 310–342.

[2] N. Bourbaki, Algèbre Commutative, Elements de Math. Vol. 30 (Hermann, Paris, 1964).

[3] M. Nagata, A theorem on valuation rings and its applications, Nagoya Math. J. 29 (1967), 85–91.

[4] J. Ohm, Some counterexamples related to integral closure in $D[[X]]$, Trans. Amer. Math. Soc. 122 (1966) 321–333.

[5] J. Ohm, Simple transcendental extensions of valued fields, J. Math. Kyoto Univ., to appear.